

Radiation reaction for multipole moments

P.O. Kazinski*

Department of Physics, Tomsk State University, Tomsk, 634050 Russia

(Dated: April 28, 2006)

We propose a Poincaré-invariant description for the effective dynamics of systems of charged particles by means of intrinsic multipole moments. To achieve this goal we study the effective dynamics of such systems within two frameworks – the particle itself and hydrodynamical one. We give a relativistic-invariant definition for the intrinsic multipole moments both pointlike and extended relativistic objects. Within the hydrodynamical framework we suggest a covariant action functional for a perfect fluid with pressure. In the case of a relativistic charged dust we prove the equivalence of the particle approach to the hydrodynamical one to the problem of radiation reaction for multipoles. As the particular example of a general procedure we obtain the effective model for a neutral system of charged particles with dipole moment.

PACS numbers: 03.50.De, 47.75.+f, 52.27.Ny, 04.20.Fy

I. INTRODUCTION

We address this paper to the problem of the construction in the classical electrodynamics framework of an effective model describing in some approximation the dynamics of an extended object consisting of charged particles. The word effective means that we take into account the self-interaction or radiation reaction force exerting on such a system and thereby eliminate from dynamical equations the electromagnetic field produced by charged particles.

The simplest models of this kind are the model of a point charged particle obeying the Lorentz-Dirac equations [1, 2] and its generalizations to curved or higher dimensional space-times [3, 4, 5]. Other models, which can be regarded as ad hoc effective models of this kind, emerge at the classical description of charged spinning particles [6, 7, 8, 9, 10, 11]. In [12] the effective model is formulated and investigated for a high-current beam of charged particles taking into account the leading self-interaction correction. In the paper [13] the effective equations of motion are obtained for the center of mass of a rigid charged body of arbitrary shape at the pointlike limit. As expected they are the Lorentz-Dirac equations. A nonrelativistic approach ($1/c$ expansion) to the effective dynamics of a system of charged particles considering the radiation reaction due to electric (magnetic) dipole and quadrupole radiation can be found in [14, 15].

In the present paper we obtain a Poincaré-invariant effective model for a system of charged point particles as the model of a point particle with some internal degrees of freedom like the intrinsic dipole moment, the intrinsic magnetic moment etc. Briefly, the procedure is as follows. We solve the Maxwell equations for arbitrary worldlines of constituent charged particles. Then we find the mean electromagnetic field in a small neighborhood of the system under consideration by averaging over all

its small-scale fluctuations¹ and substitute this field into the expression for the Lorentz force. Using the obtained system of equations we derive the equations of motion for the center of mass of the object and the evolution of intrinsic multipole moments. Neglecting terms of higher order of smallness we break the chain of equations and arrive at the closed system of equations of motion for the effective model. These steps are scrutinized in Sections II and III.

In Section II we also give a Poincaré-invariant definition of the intrinsic multipole moments. Besides, as the particular case of a general construction the effective model for a neutral pointlike system of charged particles is obtained at the end of Section III, where we estimate the energy losses of such a system and investigate its free dynamics.

In Section IV we consider an alternative approach to the description of systems of charged particles. Namely, we pass from the particle framework to hydrodynamical one and suggest an obvious generalization of the standard action functional for particles to the hydrodynamical case. This action is found to describe a relativistic dust. Then we generalize the definition of intrinsic multipole moments to the hydrodynamical approach and prove the equivalence of particle and hydrodynamical approaches to the problem of radiation reaction for multipole moments.

In addition, in Section IV we generalize the action functional for a relativistic dust to a relativistic perfect fluid with pressure and define in a Poincaré-invariant manner the intrinsic multipole moments for extended objects (charged fluids or systems of charged particles) approximated by branes, i.e. we give a relativistic definition of the linear density of dipole moment for a string or the surface density of quadrupole moment for a membrane

*Electronic address: kpo@phys.tsu.ru

¹ The averaging procedure is realized as the regularization of self-force and analogous to averaging procedures applicable in the mean field and renormalization group methods (see, e.g., [16]).

etc. In concluding section we summarize the main results and outline the prospects for further investigations.

II. EQUATIONS OF MOTION AND MULTIPOLES

In this section we recall some basic formulas concerning classical electrodynamics of many-particle systems and define multipoles for such systems in a Poincaré-invariant manner.

Let $\mathbb{R}^{3,1}$ be 4-dimensional Minkowski space with coordinates $\{x^\mu\}$, $\mu = 0, 1, 2, 3$, and signature $(+, -, -, -)$. In the space-time given a system of N electrically charged point particles with trajectories $x_a(\tau_a)$, $a = 1, \dots, N$. The dynamics of the system in question are governed by the action functional²

$$S[x_a(\tau_a), A(x)] = \sum_{a=1}^N \left[-m_a \int d\tau_a \sqrt{\dot{x}_a^2} - \int d^4x A_\mu j_a^\mu \right] - \frac{1}{16\pi} \int d^4x F_{\mu\nu} F^{\mu\nu}, \quad (1)$$

where $F_{\mu\nu} = \partial_{[\mu} A_{\nu]}$ is the strength tensor of the electromagnetic field, overdots denote derivatives with respect to the parameter on the worldline and the electric currents of point particles are

$$j_a^\mu(x) = e_a \int d\tau_a \delta^4(x - x_a(\tau_a)) \dot{x}_a^\mu(\tau_a). \quad (2)$$

The equations of motion for the action functional (1) constitute the system of Maxwell-Lorentz equations and in the Lorentz gauge $\partial^\mu A_\mu = 0$ are given by

$$m_a \frac{d}{d\tau_a} \left[\frac{\dot{x}_a^\mu}{\sqrt{\dot{x}_a^2}} \right] = e_a F_{\mu\nu}^\nu(x_a) \dot{x}_a^\mu, \quad \square A^\mu = 4\pi \sum_{a=1}^N j_a^\mu. \quad (3)$$

To obtain the effective model for a system of charged point particles we, first of all, should solve the Maxwell equations for arbitrary worldlines $x_a^\mu(\tau_a)$. The casual solution to the Maxwell equations can be constructed by the use of retarded Green's function $G(x)$ associated with the d'Alembert operator [17]:

$$G(x) = \frac{\theta(x^0)}{2\pi} \delta(x^2). \quad (4)$$

Thus the Liénard-Wiechert potentials

$$A^\mu(x) = 2 \sum_{a=1}^N e_a \int d\tau_a \theta(x^0 - x_a^0(\tau_a)) \times \delta((x - x_a(\tau_a))^2) \dot{x}_a^\mu(\tau_a) \quad (5)$$

give the solution to the Maxwell equations. They are interpreted as the electromagnetic field created by the system of charged particles. A solution of the homogeneous Maxwell equations, which can be added to the Liénard-Wiechert potentials, is regarded as an external field.

For our purposes it is useful to parametrize the trajectories of particles in the following way: let $z^\mu(\tau)$ be a naturally parametrized worldline in Minkowski space, then we parametrize the trajectories $x_a^\mu(\tau_a)$ by the parameter τ and claim that

$$\dot{z}_\rho(\tau) \xi_a^\rho(\tau) = 0, \quad \dot{z}^2(\tau) = 1, \quad (6)$$

where $\xi_a^\mu(\tau) = x_a^\mu(\tau) - z^\mu(\tau)$. The gauge (6) properly fixes parametrizations on the worldlines under the condition

$$\dot{z}_\rho \dot{x}_a^\rho > 0, \quad (7)$$

which is obviously satisfied. The worldline $z^\mu(\tau)$ is defined by the requirement

$$\sum_{a=1}^N m_a \xi_a^\mu(\tau) = 0, \quad (8)$$

and we call it as the center of mass.

By the intrinsic electric and magnetic multipole moments of the system of point charged particles we understand irreducible components of the tensors³

$$\sum_{a=1}^N e_a \xi_{\mu_1}^a \dots \xi_{\mu_n}^a, \quad \sum_{a=1}^N e_a \xi_{\mu_1}^a \dots \xi_{[\mu_{n-1}}^a \dot{\xi}_{\rho]}^a \text{pr}_{\mu_n}^\rho, \quad (9)$$

respectively, where $\text{pr}_\mu^\nu = \delta_\mu^\nu - \dot{z}^\nu \dot{z}_\mu$. Evidently, they are orthogonal to \dot{z}^μ . For example, the Liénard-Wiechert potentials (5) can be rewritten in terms of the multipole moments as

$$A_\mu(x) = q D \dot{z}_\mu + D [d_\mu - n_\rho \dot{z}_\mu^\rho] + \frac{1}{2} D [Q D \dot{z}_\mu + X^\rho Q_{\rho\sigma} X^\sigma D^2 \dot{z}_\mu + S_{\mu\nu} n^\nu + (\dot{z}^\rho - n^\rho) \dot{Q}_{\rho\sigma} (\delta_\mu^\sigma - n^\sigma \dot{z}_\mu)] + O(\xi^3), \quad (10)$$

where one must assign $\tau = \tau_{ret}$ after all differentiations. We have introduced the notations

$$X_\mu = x_\mu - z_\mu(\tau), \quad R = \dot{z}_\rho X^\rho, \quad n_\mu = \frac{X_\mu}{R}, \quad (11)$$

$$D = \frac{d}{R d\tau}, \quad X^2|_{\tau=\tau_{ret}} = 0, \quad X_0|_{\tau=\tau_{ret}} > 0,$$

² Hereinafter we use the system of units in which $c = 1$. The Greek indices are raised and lowered using the Minkowski metric $\eta_{\mu\nu}$ on $\mathbb{R}^{3,1}$. Square (round) brackets at pair indices denote antisymmetrization (symmetrization) without one half.

³ Usually electric multipole moments are defined as traceless tensors, since at large distances in stationary case the electromagnetic field of a charged object depends only on their traceless parts. But in the dynamical case the whole tensors should be employed.

and the multipole moments

$$\begin{aligned} q &= \sum_{a=1}^N e_a, & d_\mu &= \sum_{a=1}^N e_a \xi_\mu^a, \\ Q_{\mu\nu} &= \sum_{a=1}^N e_a \xi_\mu^a \xi_\nu^a, & S_{\mu\nu} &= \sum_{a=1}^N e_a \xi_{[\mu}^a \dot{\xi}_{\rho]}^a \text{pr}_\nu^\rho, \end{aligned} \quad (12)$$

which are the total charge, intrinsic dipole, quadrupole and magnetic moments respectively. Besides Q is the trace of the quadrupole moment. Higher terms in the expansion (10), which we denote as $O(\xi^3)$, can be expressed in terms of the multipole moments (9) as well.

In particular, when all the multipole moments are constant and the system moves freely we obtain a relativistic generalization of the well known expression [14]:

$$A_\mu(x) = \frac{q}{R} \dot{z}_\mu - \frac{n_\rho d^\rho}{R^2} \dot{z}_\mu + \frac{3n^\rho \bar{Q}_{\rho\sigma} n^\sigma \dot{z}_\mu + S_{\mu\nu} n^\nu}{2R^3} + O(\xi^3), \quad (13)$$

where $\bar{Q}_{\mu\nu} = Q_{\mu\nu} - \frac{1}{3} \text{pr}_{\mu\nu} Q$.

III. EFFECTIVE DYNAMICS OF MULTIPOLE MOMENTS

In this section we substitute the Liénard-Wiechert potentials into the Lorentz force and regularize the resulting ill-defined expression. After the regularization we obtain an infinite series in a regularization parameter and ξ_μ^a . To truncate this series we impose constraints on characteristic scales of the charged object under consideration. Thereby we derive the equations of motion for the effective model associated with a system of point charged particles. In particular, we obtain the effective model for a neutral pointlike object with intrinsic dipole moment, estimate the energy losses of such a system and describe its free dynamics.

The effective equations of motion of charged particles at lower orders in ξ_μ^a look like

$$\begin{aligned} m_a \frac{d}{d\tau} \left[\dot{z}_\mu + \dot{\xi}_\rho^a \text{pr}_\mu^\rho \right] &= e_a F_{\mu\nu}^{rr}(x_a(\tau)) \dot{x}_a^\nu \\ &+ e_a F_{\mu\nu}(\dot{z}^\nu + \dot{\xi}_a^\nu) + e_a \partial_\rho F_{\mu\nu} \xi_a^\rho \dot{z}^\nu + \dots, \end{aligned} \quad (14)$$

where $F_{\mu\nu}$ is the strength of the external electromagnetic field taken at the point $z^\mu(\tau)$. As one can see the linear order in ξ of particle's momentum is orthogonal to \dot{z}^μ , that is the reason that we define the intrinsic magnetic multipole moments as projected by pr_ν^μ . The field strength $F_{\mu\nu}^{rr}$ is constructed from the potentials (5) and is given by

$$\begin{aligned} F_{\mu\nu}^{rr}(x_a(\tau)) &= 4 \sum_{b=1}^N e_b \int ds \theta(X^0 + \xi_{ab}^0) \\ &\times \delta'((X + \xi^{ab})^2) (X_{[\mu} + \xi_{[\mu}^{ab} (\dot{z}_{\nu]} + \dot{\xi}_{\nu]}^b), \end{aligned} \quad (15)$$

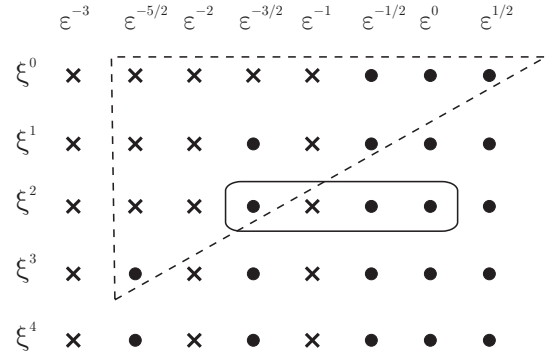


FIG. 1: The part of an infinite lattice depicting contributions of the integral (15). The vanishing terms are denoted by crosses. They are the terms at negative integer powers of ε and the terms at $\xi^k \varepsilon^{-\frac{l+3}{2}}$, $k \leq l$. The dotted triangle singles out the contributions, which we take into account for a charged system meeting (19). Three marked dots depict contributions of the self-force to the equations of motion for the center of mass of a neutral pointlike object to the accuracy of the first radiation correction.

where $X_\mu = z_\mu(\tau) - z_\mu(s)$ and $\xi_\mu^{ab} = \xi_\mu^a(\tau) - \xi_\mu^b(s)$. Now we expand the integrand of (15) in powers of ξ and arrive at

$$\begin{aligned} &\delta'(X^2) (X_{[\mu} + \xi_{[\mu}^{ab} \dot{x}_{\nu]}^b + \delta''(X^2) ((\xi^{ab})^2 + 2X^\rho \xi_\rho^{ab}) \\ &\times (\xi_{[\mu}^{ab} + X_{[\mu} \dot{x}_{\nu]}^b + 2\delta'''(X^2) (\xi_\rho^{ab} X^\rho)^2 (\xi_{[\mu}^{ab} + X_{[\mu} \dot{x}_{\nu]}^b) + \dots \end{aligned} \quad (16)$$

Here dots denote negligible in our approximation terms (see below).

After integration every term in the series (16) gives rise to infinities, because the δ -function and its derivatives are ill-defined at the vertex of light-cone (see, e.g., [18] and discussions in [5, 19, 20]). To cure this problem we have to regularize the integrals, i.e. to represent them as sequences of converging integrals. We apply the so-called “point-splitting” regularization of the δ -function and its derivatives which lies in subtraction of a positive number ε (the regularization parameter) from their arguments. Under this procedure the support of the δ -function transforms into a hyperboloid which is a smooth manifold. That is why the integrals become converging.

In fact, the regularization of the δ -function is the regularization of the Green function (4). Therefore the physical meaning of the regularization procedure consists in averaging over all small-scale fluctuations of the electromagnetic field produced by the system of charged particles up to the scale ε . That is to say the regularization parameter is a characteristic scale of fluctuations of the electromagnetic field of the charged object.

A useful mathematical framework for handling integrals with integrands like (16) is elaborated in [20] and we do not enlarge on it here. The structure of contributions of the integral (15) is as shown in Fig. 1. All the terms at half-integer powers of ε are Lagrangian [20], i.e. can be obtained by varying an effective action, which in

turn is obtained from the action functional (1) by substituting the Liénard-Wiechert potentials (5) in it and applying the regularization procedure. The terms at integer powers of ε are responsible for radiation losses and not restored by the effective action.

Let l be a characteristic scale of variations of the trajectory $z(\tau)$. Then, firstly, we average over⁴ all oscillations of the fields $\xi_\mu^a(\tau)$ the frequencies of which are greater than l^{-1} . In that case the n -th derivative of $\xi_\mu^a(\tau)$ with respect to τ is of the order ξ/l^n , where ξ is a characteristic scale of fluctuations of the variables $\xi_\mu^a(\tau)$. As the result for the integral (15) we have an infinite series in two dimensionless variables ξ/l and $\varepsilon^{1/2}/l$.

Secondly, one can see from Fig. 1 the necessary condition for the obtained asymptotic expansion makes sense is

$$\xi \ll \varepsilon^{1/2} \ll l. \quad (17)$$

Physically, the condition (17) means that we consider a charged system which is much smaller than the characteristic scale of fluctuations of the electromagnetic field (after averaging) while this characteristic scale is much smaller than the characteristic scale of variations of the trajectory $z(\tau)$.

Thirdly, in the lack of any a priori data on the system

⁴ It is necessary so as to the time-scale of variations of the fields $\xi_\mu^a(\tau)$ would be larger than $\varepsilon^{1/2}$. Otherwise the terms at higher

of charged particles in question we make the assumption that there exists n such that

$$M^{(n+k)} \lesssim M^{(n)} (\xi/l)^k, \quad (18)$$

for any positive integer k . Here $M^{(n)}$ denotes a magnitude of the n -th multipole moment (9). In other words we assume that beginning from the n -th order multipole collective phenomena such as spontaneous magnetization etc. are absent in the charged object, i.e. there is no any high ordered structure in it. If that is the case then the leading contribution to the self-force is contained in the first n orders in ξ of the asymptotic expansion.

In subsequent calculations we mostly deal with the case $n \leq 2$, moreover we take

$$\xi/l \ll (\varepsilon^{1/2}/l)^{\frac{4}{3}}, \quad \varepsilon/l^2 \lesssim \xi/l. \quad (19)$$

These conditions mean that we consider a charged object which is much smaller than $\varepsilon^{1/2}$, but is not so small to be pointlike. Some of the above requirements could be relaxed, however it deserves individual investigations.

So, neglecting the terms of higher order of smallness we obtain

powers of ε make greater contributions than at lower ones.

$$\begin{aligned} F_{\mu\nu}^{rr}(x_a(\tau)) = & -4 \sum_{b=1}^N e_b \left\{ \frac{3\varepsilon^{-\frac{5}{2}}}{8} (\xi^{ba})^2 \xi_{[\mu}^b \dot{z}_{\nu]} + \frac{\varepsilon^{-\frac{3}{2}}}{16} \left[4\xi_{[\mu}^{ba} \dot{x}_{\nu]}^b + (\xi^{ba})^2 \ddot{z}_{[\mu} \dot{z}_{\nu]} - 2(\ddot{z}^\rho \xi_\rho^{ba} + 2\dot{z}^\rho \dot{\xi}_\rho^b) \xi_{[\mu}^{ba} \dot{z}_{\nu]} \right] \right. \\ & + \frac{\varepsilon^{-\frac{1}{2}}}{16} \left[(2 - 3\ddot{z}^\rho \xi_\rho^{ba} - 6\dot{z}^\rho \dot{\xi}_\rho^b) \ddot{z}_{[\mu} \dot{z}_{\nu]} + 2\ddot{z}_{[\mu} \dot{\xi}_{\nu]}^b + 2\ddot{\xi}_{[\mu}^b \dot{z}_{\nu]} - 2\xi_{[\mu}^{ba} \ddot{z}_{\nu]} - \frac{\ddot{z}^2}{2} \xi_{[\mu}^{ba} \dot{z}_{\nu]} \right] \\ & \left. - \frac{1}{6} \ddot{z}_{[\mu} \dot{z}_{\nu]} + \frac{\varepsilon^{\frac{1}{2}}}{32} \left[3\ddot{z}_{[\mu}^{(4)} \dot{z}_{\nu]} + 2\ddot{z}_{[\mu} \ddot{z}_{\nu]} + \frac{5}{2} \ddot{z}^2 \ddot{z}_{[\mu} \dot{z}_{\nu]} \right] \right\}, \quad (20) \end{aligned}$$

where $\xi_\mu^{ba} = \xi_\mu^b(\tau) - \xi_\mu^a(\tau)$. Hence in our approximation the average radiation reaction force acting on the charged point particle a of the system at issue reads

$$\begin{aligned} e_a F_{\mu\nu}^{rr}(x_a(\tau)) \dot{x}_a^\nu = & - \sum_{b=1}^N e_a e_b \left\{ \frac{3\varepsilon^{-\frac{5}{2}}}{2} (\xi^{ba})^2 \xi_\mu^{ba} + \frac{\varepsilon^{-\frac{3}{2}}}{4} \left[(4 + 2\dot{z}^\rho (\dot{\xi}_\rho^b + \dot{\xi}_\rho^a)) \xi_\mu^{ba} + (\xi^{ba})^2 \ddot{z}_\mu - 4\dot{\xi}^{a\rho} \xi_\rho^{ba} \dot{z}_\mu \right] \right. \\ & + \frac{\varepsilon^{-\frac{1}{2}}}{4} \left[(2 - \dot{z}^\rho (\dot{\xi}_\rho^b + \dot{\xi}_\rho^a)) \ddot{z}_\mu - 2(\ddot{z}^\rho \dot{\xi}_\rho^a + \dot{z}^\rho \ddot{\xi}_\rho^b) \dot{z}_\mu + 2\ddot{\xi}_\mu^b + \frac{3}{2} \ddot{z}^2 \xi_\mu^{ba} \right] \\ & \left. - \frac{2}{3} [\ddot{z}_\mu + \ddot{z}^2 \dot{z}_\mu] + \frac{3\varepsilon^{\frac{1}{2}}}{8} \left[\ddot{z}_\mu^{(4)} + \frac{3}{2} \ddot{z}^2 \ddot{z}_\mu + 3\ddot{z}^\rho \ddot{z}_\rho \dot{z}_\mu \right] \right\}. \quad (21) \end{aligned}$$

Numerical coefficients at powers of the regularization parameter depend on the regularization scheme applied. Different powers of the regularization parameter should be regarded as independent constants to be taken from

an experiment. The averaging procedure, which we have used, gives only the relations between orders of magnitudes of these constants.

It is useful to introduce physical masses of particles

$$\tilde{m}_a = m_a + \frac{e_a q}{2} \varepsilon^{-\frac{1}{2}}, \quad (22)$$

and redefine $z(\tau)$ by the requirement (8) with respect to them. Then the equations of motion for the center of mass look like

$$\begin{aligned} (\tilde{M} + G)\ddot{z}_\mu &= F_\mu^{LD} + F_\mu^{(6)} + F_{\mu\nu}(q\dot{z}^\nu + \dot{d}^\nu) \\ &\quad + F_{\rho\sigma}\dot{z}^\rho\dot{z}^\sigma\dot{z}_\mu + d^\rho\partial_\rho F_{\mu\nu}\dot{z}^\nu, \\ \dot{G} &= F_{\rho\sigma}\dot{z}^\rho\dot{d}^\sigma, \quad \tilde{M} = \sum_{a=1}^N \tilde{m}_a, \quad G = \frac{\varepsilon^{-\frac{3}{2}}}{2}(qQ - d^2), \end{aligned} \quad (23)$$

where $F_\mu^{LD} = \frac{2}{3}q^2[\ddot{z}_\mu + \ddot{z}^2\dot{z}_\mu]$ is the Lorentz-Dirac force and

$$F_\mu^{(6)} = -\frac{3\varepsilon^{\frac{1}{2}}}{8}q^2 \left[\overset{(4)}{z}_\mu + \frac{3}{2}\ddot{z}^2\dot{z}_\mu + 3\ddot{z}^\rho\ddot{z}_\rho\dot{z}_\mu \right]. \quad (24)$$

Formally, the rigid term (24) is nothing but the counter term which, in addition to an ordinary mass renormalization, compensates the divergencies emerging in six-dimensional classical electrodynamics of a point charged particle [4, 5]. Such rigid terms have long been known for nonrelativistic models of extended charged particles in the four-dimensional space-time (see, e.g., [21] and references therein).

From (22) and the condition of positiveness of the total mass $\sum_{a=1}^N m_a$ we recover

$$q^2/2\tilde{M} < \varepsilon^{1/2}. \quad (25)$$

Furthermore, in what follows we imply the severer estimation

$$q^2/\tilde{M} \lesssim \xi, \quad (26)$$

which nevertheless is the reasonable one.

In order to close the system (23) and obtain the effective model for a charged object we have to know the time evolution of the dipole moment. To this end we make use of Eqs. (14), (21) again. At lower orders they are

$$\begin{aligned} \tilde{m}_a \frac{d}{d\tau} \left[\dot{z}_\mu + \dot{\xi}_\rho^a \text{pr}_\rho^\mu \right] &= e_a \varepsilon^{-\frac{3}{2}} (q\xi_\mu^a - d_\mu) \\ &\quad + e_a F_{\mu\nu}(\dot{z}^\nu + \dot{\xi}_a^\nu) + e_a \xi_a^\rho \partial_\rho F_{\mu\nu} \dot{z}^\nu. \end{aligned} \quad (27)$$

The equations of motion describe the dynamics of mechanical moments therefore we should determine relations between electromagnetic and mechanical moments.

The simplest way to do it is to pick out different species of particles.

We call particles to be of the same species iff they have an equal ratio $\lambda = e/\tilde{m}$. Let the system be made of K species of charged particles with charge to mass ratios λ_s , $s = 1, \dots, K$. Then summing in Eqs. (27) over particles of the same species we obtain

$$\begin{aligned} \frac{d}{d\tau} \left[\dot{d}_\mu^s + (q_s + \ddot{z}^\rho d_\rho^s) \dot{z}_\mu \right] &= \lambda_s \varepsilon^{-\frac{3}{2}} (q d_\mu^s - q_s d_\mu) \\ &\quad + \lambda_s F_{\mu\nu} (q_s \dot{z}^\nu + \dot{d}_s^\nu) + \lambda_s d_s^\rho \partial_\rho F_{\mu\nu} \dot{z}^\nu, \end{aligned} \quad (28)$$

where q_s and d_μ^s are the total charge and dipole moment corresponding to the species s . Equations (23) and (28) must be solved perturbatively starting from the Lorentz equations

$$\tilde{M}\ddot{z}_\mu = qF_{\mu\nu}\dot{z}^\nu, \quad (29)$$

and taking into account the estimation (26). In doing so we eliminate all the higher derivatives of $z^\mu(\tau)$.

While the total charge of the object under consideration is sufficiently small, i.e. $q^2/\tilde{M} \ll \xi$, in substituting to (28) one can disregard the Lorentz-Dirac force, G -term and the rigid term in the equations of motion for the center of mass (23) and rewrite the equations (28) as

$$\begin{aligned} \ddot{d}_\mu^s + \frac{q}{\tilde{M}} \frac{d}{d\tau} (d_s^\rho F_{\rho\sigma} \dot{z}^\sigma) \dot{z}_\mu &= \lambda_s \varepsilon^{-\frac{3}{2}} (q d_\mu^s - q_s d_\mu) \\ &\quad + \left[q_s \left(\lambda_s - \frac{q}{\tilde{M}} \right) - \frac{q^2}{\tilde{M}^2} d_s^\rho F_{\rho\sigma} \dot{z}^\sigma \right] F_{\mu\nu} \dot{z}^\nu \\ &\quad + F_{\mu\nu} \left(\lambda_s \dot{d}_s^\nu - \frac{q_s}{\tilde{M}} \dot{d}^\nu \right) + \left(\lambda_s d_s^\rho - \frac{q_s}{\tilde{M}} d^\rho \right) \partial_\rho F_{\mu\nu} \dot{z}^\nu. \end{aligned} \quad (30)$$

In the case of $q^2/\tilde{M} \approx \xi$ some additional terms appear.

Thus we obtain a closed system of equations describing in our approximation the evolution of a charged object. To put it in another way, singled out essential degrees of freedom and averaged over or neglected the rest of ones we derive the aforementioned effective model.

The equations (23) and (28), (30) can be regarded as the starting point for further perturbation theory in ξ to take into account higher multipoles effects. For example, these last inevitably appear when one needs to consider the self-interaction corrections to (28) to the same accuracy as in Eqs. (23). Consequently, it is worthwhile to obtain the time evolution for the second order in ξ multipoles. Similarly to the above considerations from Eqs. (27) we find at lower orders

$$\begin{aligned}
\lambda_s \dot{T}_{\mu\nu}^s + \text{pr}_{(\mu}^\rho d_{\rho}^s \ddot{z}_{\nu)} &= \lambda_s \left\{ \varepsilon^{-\frac{3}{2}} \left[q \left(\dot{Q}_{\mu\nu}^s - \dot{Q}_{(\mu\rho}^s \bar{\text{pr}}_{\nu)}^\rho \right) - \text{pr}_{(\mu}^\rho d_{\rho}^s d_{\nu)} \right] - \lambda_s T_{(\mu}^s F_{\rho\nu)} \right. \\
&\quad \left. + \left[\text{pr}_{(\mu}^\rho d_{\rho}^s - \frac{1}{2} \ddot{z}^\rho \left(\dot{Q}_{\rho\sigma}^s \text{pr}_{(\mu}^\sigma + S_{\rho(\mu)}^s \right) \right] F_{\nu)\sigma} \dot{z}^\sigma + \frac{1}{2} \left(\dot{Q}_{(\mu\sigma}^s \text{pr}_{\rho}^\sigma - \bar{\text{pr}}_{(\mu}^\sigma \dot{Q}_{\sigma\rho}^s - S_{(\mu\rho)}^s \right) \partial^\rho F_{\nu)\sigma} \dot{z}^\sigma \right\}, \\
\frac{d}{d\tau} \left[\dot{Q}_{\mu\nu}^s + Q_{(\mu\rho}^s \ddot{z}^\rho \dot{z}_{\nu)} \right] + d_{(\mu}^s \ddot{z}_{\nu)} &+ \frac{1}{2} \dot{z}_{(\mu} \ddot{z}^\rho \left[\dot{Q}_{\rho\sigma}^s \text{pr}_{\nu)}^\sigma + S_{\rho\nu)}^s \right] = \lambda_s \left\{ \varepsilon^{-\frac{3}{2}} \left[2q Q_{\mu\nu}^s - d_{(\mu}^s d_{\nu)} \right] \right. \\
&\quad \left. + 2T_{\mu\nu}^s + d_{(\mu}^s F_{\nu)\rho} \dot{z}^\rho - \frac{1}{2} \left[\text{pr}_{(\mu}^\sigma \dot{Q}_{\sigma\rho}^s + \dot{Q}_{(\mu\sigma}^s \bar{\text{pr}}_{\rho}^\sigma + S_{(\mu\rho)}^s \right] F_{\nu)}^\rho + Q_{(\mu\rho}^s \partial^\rho F_{\nu)\sigma} \dot{z}^\sigma \right\}, \\
\dot{S}_{\mu\nu}^s + d_{[\mu}^s \ddot{z}_{\nu]} &+ \frac{1}{2} \dot{z}_{[\mu} \ddot{z}^\rho \left[\dot{Q}_{\rho\sigma}^s \text{pr}_{\nu]}^\sigma + S_{\rho\nu]}^s \right] = \lambda_s \left\{ \varepsilon^{-\frac{3}{2}} d_{[\mu}^s d_{\nu]} + d_{[\mu}^s F_{\nu]\rho} \dot{z}^\rho \right. \\
&\quad \left. - \frac{1}{2} \left[\text{pr}_{[\mu}^\sigma \dot{Q}_{\sigma\rho}^s + \dot{Q}_{[\mu\sigma}^s \bar{\text{pr}}_{\rho}^\sigma + S_{[\mu\rho]}^s \right] F_{\nu]}^\rho + Q_{[\mu\rho}^s \partial^\rho F_{\nu]\sigma} \dot{z}^\sigma \right\}, \quad (31)
\end{aligned}$$

where $\bar{\text{pr}}_\mu^\nu = \delta_\mu^\nu - \text{pr}_\mu^\nu$, and $T_{\mu\nu}^s = \sum \tilde{m}_a \text{pr}_\mu^\rho \text{pr}_\nu^\sigma \xi_\rho^a \dot{\xi}_\sigma^a$ can be interpreted as the intrinsic stress (momentum flux) tensor⁵ corresponding to the species s . As all the intrinsic quantities we have defined it is orthogonal to \dot{z}^μ . Again we arrive at the closed system of equations of motion

⁵ Strictly speaking, the stress tensor is $\int d\tau \delta^4(x - z(\tau)) T_{\mu\nu}(\tau)$.
⁶ This case is of importance since it can be considered as the base for perturbation theory in the deviation of the ratios λ_i from

(23), (28), (31) for the effective model.

Provided that the charged object is composed of particles of one species⁶ the intrinsic dipole moment vanishes due to (8) and the system of equations (31) looks like their mean value.

$$\begin{aligned}
\lambda \dot{T}_{\mu\nu} &= \lambda \left\{ q \varepsilon^{-\frac{3}{2}} \left(\dot{Q}_{\mu\nu} - \dot{Q}_{(\mu\rho} \bar{\text{pr}}_{\nu)}^\rho \right) - \lambda T_{(\mu}^\rho F_{\rho\nu)} \right. \\
&\quad \left. - \frac{1}{2} \ddot{z}^\rho \left(\dot{Q}_{\rho\sigma} \text{pr}_{(\mu}^\sigma + S_{\rho(\mu)} \right) F_{\nu)\sigma} \dot{z}^\sigma + \frac{1}{2} \left(\dot{Q}_{(\mu\sigma} \text{pr}_{\rho}^\sigma - \bar{\text{pr}}_{(\mu}^\sigma \dot{Q}_{\sigma\rho} - S_{(\mu\rho)} \right) \partial^\rho F_{\nu)\sigma} \dot{z}^\sigma \right\}, \\
\frac{d}{d\tau} \left[\dot{Q}_{\mu\nu} + Q_{(\mu\rho} \ddot{z}^\rho \dot{z}_{\nu)} \right] + \frac{1}{2} \dot{z}_{(\mu} \ddot{z}^\rho \left[\dot{Q}_{\rho\sigma} \text{pr}_{\nu)}^\sigma + S_{\rho\nu)} \right] &= \lambda \left\{ 2q \varepsilon^{-\frac{3}{2}} Q_{\mu\nu} + 2T_{\mu\nu} \right. \\
&\quad \left. - \frac{1}{2} \left[\text{pr}_{(\mu}^\sigma \dot{Q}_{\sigma\rho} + \dot{Q}_{(\mu\sigma} \bar{\text{pr}}_{\rho}^\sigma + S_{(\mu\rho)} \right] F_{\nu)}^\rho + Q_{(\mu\rho} \partial^\rho F_{\nu)\sigma} \dot{z}^\sigma \right\}, \\
\dot{S}_{\mu\nu} + \frac{1}{2} \dot{z}_{[\mu} \ddot{z}_{\nu]} \left[\dot{Q}_{\rho\sigma} \text{pr}_{\nu]}^\sigma + S_{\rho\nu]} \right] &= -\frac{\lambda}{2} \left[\text{pr}_{[\mu}^\sigma \dot{Q}_{\sigma\rho} + \dot{Q}_{[\mu\sigma} \bar{\text{pr}}_{\rho}^\sigma + S_{[\mu\rho]} \right] F_{\nu]}^\rho + \lambda Q_{[\mu\rho} \partial^\rho F_{\nu]\sigma} \dot{z}^\sigma. \quad (32)
\end{aligned}$$

The equations of motion for the center of mass (23) are modified into

$$\begin{aligned}
(\tilde{M} + G) \ddot{z}_\mu &= F_\mu^{LD} + F_\mu^{(6)} + q F_{\mu\nu} \dot{z}^\nu + \frac{1}{2} Q^{\rho\sigma} \partial_{\rho\sigma} F_{\mu\nu} \dot{z}^\nu \\
&\quad + \frac{1}{2} (\text{pr}^{\rho\sigma} \dot{Q}_\sigma^\nu + \dot{Q}^{\rho\sigma} \bar{\text{pr}}_\sigma^\nu + S^{\rho\nu}) \partial_\rho F_{\mu\nu}, \\
\dot{G} &= \frac{1}{2} \dot{Q}^{\rho\nu} \partial_\rho F_{\mu\nu} \dot{z}^\mu + \frac{1}{2} (S^{\rho\nu} - \bar{\text{pr}}_\sigma^\rho \dot{Q}^{\sigma\nu}) \dot{F}_{\rho\nu}, \quad (33)
\end{aligned}$$

where we add the first corrections in ξ to the external force acting on the charged object as a whole.

For instance, for the uniform external field $F_{\mu\nu} = \text{const}$ and constant quadrupole moment $Q_{\mu\nu} = \text{const}$ the last equation of the system (32) is nothing but the Bargmann-

Michel-Telegdi equation [6, 7, 8, 22, 23]. The rest of Eqs. (32) are equivalent to

$$\lambda T_{\mu\nu} = \frac{\lambda}{4} S_{(\mu\rho} F_{\nu)}^\rho + \frac{1}{4} \dot{z}_{(\mu} \ddot{z}_{\nu)} S_{\rho\nu)} - \lambda q \varepsilon^{-\frac{3}{2}} Q_{\mu\nu}. \quad (34)$$

The trace of this equation gives rise to the virial theorem in our approximation. Notice as the quadrupole moment is constant there is an inertial frame in which $Q_{\mu\nu}$ becomes diagonal for all times. The condition $Q_{\mu\nu} \dot{z}^\nu = 0$ implies either this is the comoving frame and $\ddot{z}_\mu = 0$ or at least one of the space-like diagonal elements is zero, i.e. it is sufficiently small. If only one diagonal element is zero (the case of a thin-plate object) then the vector of intrinsic magnetic moment is parallel to the respective axis. If two or three elements vanish then $S_{\mu\nu} = 0$.

As we have already noted the terms in Eqs. (21) and (23) at half-integer powers of ε are Lagrangian. In order to simplify the Lagrangian we write down it in the natural parametrization and keep only the terms which are at most linear in the transverse gauge (6):

$$L = \sum_{a,b=1}^N e_a e_b \left\{ \frac{3\varepsilon^{-\frac{5}{2}}}{16} (\xi^{ba})^4 \right. \\ \left. + \frac{\varepsilon^{-\frac{3}{2}}}{4} [(\xi^{ba})^2 (1 + \dot{z}^\rho \xi_\rho^b) - 2\dot{\xi}^{b\rho} \xi_\rho^{ba} \dot{z}^\sigma \xi_\sigma^{ba}] \right. \\ \left. + \frac{\varepsilon^{-\frac{1}{2}}}{32} [16 + 16\dot{z}^\rho \xi_\rho^a + 8\dot{\xi}^{a\rho} \xi_\rho^b + 3(\xi^{ba})^2 \ddot{z}^2 - 8\dot{z}^\rho \xi_\rho^a \dot{z}^\sigma \xi_\sigma^b \right. \\ \left. + 16\dot{z}^\rho \xi_\rho^{ba} \dot{z}^\sigma \xi_\sigma^b - 2(\dot{z}^\rho \xi_\rho^{ba})^2] - \frac{3\varepsilon^{\frac{1}{2}}}{16} [\ddot{z}^2 - 3\ddot{z}^2 \dot{z}^\rho \xi_\rho^a + 2\ddot{z}^\rho \xi_\rho^a] \right\}. \quad (35)$$

Varying the effective action with the Lagrangian density (35) with respect to $\xi_\mu^a(\tau)$ we arrive at Eqs. (21), varying it with respect to $z^\mu(\tau)$ and dropping out the terms of higher order in ξ we obtaining Eqs. (23). In both cases the Lorentz-Dirac force are not of course reproduced.

To conclude this section we derive the effective equations of motion for a neutral system of charged particles at the pointlike limit, i.e. at $q = 0$ and $\xi \rightarrow 0$. Rigorously,

we consider the system for which

$$\xi/l \ll (\varepsilon^{\frac{1}{2}}/l)^5, \quad q^2 l^2 \ll d^2, \quad (36)$$

where d is a magnitude of the intrinsic dipole moment. We also assume the estimation (18) is fulfilled for $n = 1$.

In what follows we are interested in the equations of motion for the center of mass of the object in question taking into account the first correction due to radiation, i.e. to the accuracy of the leading non-Lagrangian contribution like the Lorentz-Dirac force in the case of charged particle. At Fig. 1 the required terms are pictured as three dots at the third line corresponding to the order of ξ^2 . In view of (36) other contributions are negligible.

The action for two Lagrangian terms can be evidently deduced from (35). Its Lagrangian density is equal to

$$L_{q=0} = -\frac{\varepsilon^{-\frac{3}{2}}}{2} d^2 + \frac{\varepsilon^{-\frac{1}{2}}}{16} [4d^2 - 3d^2 \ddot{z}^2 - 4(\dot{z}^\rho \dot{d}_\rho)^2 - 8\dot{z}^\rho d_\rho \ddot{z}^\sigma \dot{d}_\sigma + 2(\ddot{z}^\rho d_\rho)^2]. \quad (37)$$

Thus we have to find the non-Lagrangian term only. Tedious calculations analogous to the case of a charged system show up that the non-Lagrangian part of the self-force exerting on the system as a whole looks like

$$F_\mu^{rr} = \frac{4}{15} d^2 \ddot{z}_\mu^{(5)} + \frac{4}{3} d^\rho \dot{d}_\rho \ddot{z}_\mu^{(4)} + \frac{2}{3} \left[2d^\rho \ddot{d}_\rho + \dot{d}^2 + 2(\dot{z}^\rho \dot{d}_\rho)^2 + \ddot{z}^2 d^2 \right] \ddot{z}_\mu \\ + \frac{2}{3} \left\{ d^\rho \ddot{\ddot{d}}_\rho + \frac{d}{d\tau} \left[(\dot{z}^\rho \dot{d}_\rho)^2 + \frac{3}{2} \ddot{z}^2 d^2 \right] \right\} \ddot{z}_\mu - \frac{2}{3} \left\{ \dot{d}^\rho \ddot{\ddot{d}}_\rho - (\ddot{z}^\rho \dot{d}_\rho + \dot{z}^\rho \ddot{d}_\rho)^2 - 2\ddot{z}^\rho \dot{d}_\rho \dot{z}^\sigma \ddot{d}_\sigma \right. \\ \left. + 2\dot{z}^\rho \dot{d}_\rho (\ddot{z}^\sigma \dot{d}_\sigma - \dot{z}^\sigma \ddot{\ddot{d}}_\sigma) - \ddot{z}^2 \left[d^\rho \ddot{d}_\rho + 7(\dot{z}^\rho \dot{d}_\rho)^2 \right] - 5\ddot{z}^\rho \ddot{\ddot{z}}_\rho d^\sigma \dot{d}_\sigma + d^2 \left[\frac{2}{5} \dot{z}^\rho \ddot{\ddot{z}}_\rho^{(5)} - \ddot{z}^4 \right] \right\} \dot{z}_\mu \\ - \frac{2}{3} \left[\ddot{z}^\rho \ddot{\ddot{d}}_\rho + \ddot{\ddot{z}}^\rho \ddot{d}_\rho + \dot{z}^\rho \ddot{d}_\rho + \frac{1}{5} \dot{z}^\rho \ddot{d}_\rho + \ddot{z}^2 (\ddot{z}^\rho \dot{d}_\rho - \dot{z}^\rho \ddot{d}_\rho) \right] d_\mu \\ - \frac{2}{3} \left[\ddot{z}^\rho \dot{d}_\rho - 3\ddot{z}^2 \ddot{z}^\rho d_\rho \right] \dot{d}_\mu + \frac{2}{3} \ddot{\ddot{z}}^\rho d_\rho \ddot{d}_\mu - \frac{2}{3} \dot{z}^\rho \dot{d}_\rho \ddot{\ddot{d}}_\mu. \quad (38)$$

Gathering all the terms together we arrive at the following equations for the center of mass:

$$M \ddot{z}_\mu = F_\mu^{lagr} + F_\mu^{rr} + F_{\mu\nu} \dot{d}^\nu + d^\rho \partial_\rho F_{\mu\nu} \dot{z}^\nu, \quad (39)$$

where F_μ^{lagr} denote Lagrangian self-forces coming from (37). To the same accuracy the time evolution of the intrinsic dipole moment obeys

$$\frac{d}{d\tau} \left[\dot{d}_\mu^s + (q_s + \ddot{z}^\rho d_\rho^s) \dot{z}_\mu \right] = -\lambda_s q_s \left\{ \varepsilon^{-\frac{3}{2}} d_\mu + \frac{\varepsilon^{-\frac{1}{2}}}{4} \left(2\ddot{d}_\mu + \frac{3}{2} \ddot{z}^2 d_\mu - \dot{z}^\rho \dot{d}_\rho \ddot{z}_\mu - 2\dot{z}^\rho \ddot{d}_\rho \dot{z}_\mu \right) \right. \\ \left. + \frac{2}{3} \left[\dot{z}^\rho \dot{d}_\rho \ddot{z}_\mu + (\ddot{z}^\rho \dot{d}_\rho + 2\dot{z}^\rho \ddot{d}_\rho) \ddot{z}_\mu + (\dot{z}^\rho \ddot{\ddot{d}}_\rho + 2\ddot{z}^2 \dot{z}^\rho \dot{d}_\rho) \dot{z}_\mu - \ddot{z}^\rho \ddot{\ddot{z}}_\rho d_\mu - \ddot{z}^2 \dot{d}_\mu - \ddot{\ddot{d}}_\mu \right] \right\} \\ + \lambda_s \left[F_{\mu\nu} (q_s \dot{z}^\nu + \dot{d}_s^\nu) + d_s^\rho \partial_\rho F_{\mu\nu} \dot{z}^\nu \right]. \quad (40)$$

The contributions at the first and third lines are simply obtained from Eqs. (21), while the term at the second line is derived ab initio.

So, we derive the equations of motion for the effective model for a neutral pointlike object with dipole moment. Note that in a similar manner the equations of motion can be derived for the effective model for a neutral pointlike object with magnetic moment and vanishing dipole moment.

It is reasonable to suppose that the energy of the self-interaction is much smaller than the rest energy of particles in the absence of interaction, i.e.

$$d^2/\varepsilon^{3/2} \ll M. \quad (41)$$

In that case all the higher derivatives of $z_\mu(\tau)$ are perturbatively expressed in terms of the first ones by means of equations (39). As far as \ddot{d}_μ are concerned the similar procedure can be applied to Eqs. (40) under the assumption that $\lambda_i q_i l^2/\varepsilon^{\frac{3}{2}}$ are of the order of unity or less.

Let us extract some physical information from the huge equations (39), (40). First of all, as one can see from Eqs. (38) the total radiated power at small accelerations of the center of mass of the object is equal to

$$\mathcal{P}_\mu = \frac{2}{3} \dot{d}^\rho \ddot{d}_\rho \dot{z}_\mu, \quad (42)$$

and coincides with the known nonrelativistic expression (see, e.g., [14]) for a dipole radiation. Besides, as it follows from Eqs. (38) in the nonrelativistic limit⁷ the total radiated power is

$$\mathcal{P}_\mu = \left[\frac{2}{3} \dot{d}^\rho \ddot{d}_\rho - \frac{4}{15} \frac{d}{d\tau} (2d^\rho \ddot{d}_\rho + \dot{d}^\rho \ddot{d}_\rho) \right] \dot{z}_\mu. \quad (43)$$

Whence the average emitted energy per unit time in the lab frame amounts to $\frac{2}{3} \ddot{\mathbf{d}} \ddot{\mathbf{d}}$.

In the absence of external fields and at zero acceleration of the center of mass the equations of motion (39), (40) are reduced to⁸

$$\begin{aligned} \frac{d}{d\tau} \left[\frac{\varepsilon^{-\frac{1}{2}}}{4} \dot{d}^2 + \frac{\varepsilon^{-\frac{3}{2}}}{2} d^2 \right] &= \frac{2}{3} \dot{d}^\rho \ddot{d}_\rho, \\ \left(1 + \frac{\alpha \varepsilon^{-\frac{1}{2}}}{2} \right) \ddot{d}_\mu &= -\alpha \varepsilon^{-\frac{3}{2}} d_\mu + \frac{2\alpha}{3} \ddot{d}_\mu, \end{aligned} \quad (44)$$

respectively. In order to the system of equations (44) possesses nontrivial solutions with the characteristic scale

of variations more than or equal to l the positive quantity $\alpha = \sum_{s=1}^K \lambda_s q_s$ is supposed to meet the condition

$$\alpha^2/\varepsilon \ll \varepsilon^{\frac{1}{2}}/l \ll 1. \quad (45)$$

The system (44) can be satisfied by the second equation only provided the additional requirement

$$d^2 = \text{const} \quad (46)$$

is fulfilled.

By the general procedure we must solve Eqs. (44) perturbatively, but in our case we can do it exactly and make approximations directly in a general solution. The characteristic numbers associated to the second equation in (44) are rather complicated and we write down them in an approximate form

$$\begin{aligned} \lambda_{1,2} &\approx \pm i\omega - \gamma, & \lambda_3 &\approx \frac{\omega^2}{2\gamma}, \\ \omega^2 &= \alpha' \varepsilon^{-\frac{3}{2}}, & \gamma &= \frac{\alpha'^2}{3} \varepsilon^{-\frac{3}{2}}, & \alpha' &= \frac{2\alpha}{2 + \alpha \varepsilon^{-\frac{1}{2}}}, \end{aligned} \quad (47)$$

taking into account the estimation (45). The solution corresponding to the third characteristic number is known as runaway solution. We should drop it out since it does not meet the initial hypothesis concerning the characteristic scale of variations of the fields $\xi_\mu^a(\tau)$, i.e. $\lambda_3 \gg l^{-1}$. By the assumption (45) the damping factor γ is small, $\gamma \ll l^{-1}$, that is the reason that the solution of the second equation in the system (44)

$$\begin{aligned} d_\mu(\tau) &= [a_\mu \cos(\omega\tau) + b_\mu \sin(\omega\tau)] e^{-\gamma\tau}, \\ a^2 &= b^2, & a_\rho b^\rho &= 0, \end{aligned} \quad (48)$$

approximately satisfies the requirement (46). This solution represents a freely moving slowly rotating dipole.

IV. HYDRODYNAMICAL APPROACH

In this section we regard a Poincaré-invariant hydrodynamical approach to the description of a system of charged particles thought as a relativistic perfect fluid. We briefly reformulate basic definitions previously introduced within the particle framework and prove the equivalence of the hydrodynamical and particle approaches to the problem of radiation reaction for multipoles under some assumptions given below. In conclusion of this section we give a generalization of the notion of multipole moments to extended relativistic objects (branes).

A simple general covariant generalization of the action (1) to the hydrodynamical case can be constructed as follows. Suppose given a 3-brane N with coordinates $\{\tau^i\}$, $i = 0, 1, 2, 3$, that is embedded by a diffeomorphism $x^\mu(\tau)$ into Minkowski space $\mathbb{R}^{3,1}$. If the brane N is the space-time itself one can think about such a diffeomorphism as a general coordinate transformation in it. Let

⁷ Recall that on recovering the velocity of light every overdot contains $1/c$.

⁸ The second equation in (44) resembles the equation of motion for a damped linear oscillator. The existence of self-oscillations of charged distributions has been known long ago (see, e.g., [24]).

us introduce $\rho^i(\tau)$ and $e^i(\tau)$ that are vector densities on the brane N and describe a mass flow and an electric current respectively. Then an obvious generalization of the action (1) takes the form of⁹

$$S[x(\tau), A(x)] = - \int_N d^4\tau \sqrt{\rho^i \rho^j h_{ij}} - \int_{\mathbb{R}^{3,1}} d^4x \left[A_\mu j^\mu + \frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} \right], \quad (49)$$

where $h_{ij} = \partial_i x^\mu \partial_j x^\nu \eta_{\mu\nu}$ is the induced metric on the brane N , it being flat inasmuch as the space-time is flat. The vector densities $\rho^i(\tau)$ and $e^i(\tau)$ are supposed to vanish at spatial infinity. The electric current density on the target space (the space-time) is

$$j^\mu(x) = \int_N d^4\tau \delta^4(x - x(\tau)) e^i(\tau) \partial_i x^\mu(\tau). \quad (50)$$

Of course, the δ -function can be integrated out resulting in the Jacobian, but we leave it intact to keep an analogy with (2). In other words the matter action in (49) defines the dynamics on the infinite-dimensional group of diffeomorphisms¹⁰ $x(\tau^0, \bar{\tau})$ of the space-like hypersurfaces: $\tau^0 = \text{const}$ and its image in the space-time. Hereinafter we denote $\bar{\tau} = \{\tau^1, \tau^2, \tau^3\}$.

An invariance of the action functional (49) under gauge transformations of the electromagnetic potentials implies the charge conservation law

$$\partial_i e^i = 0. \quad (51)$$

Then the equations of motion for the matter obtained from the action (49) read

$$\partial_i \left[\frac{\rho^i \rho^j \partial_j x_\mu}{\sqrt{\rho^2}} \right] = F_{\mu\nu} e^i \partial_i x^\nu. \quad (52)$$

The matter energy-momentum tensor is¹¹

$$T_{mat}^{\mu\nu} = \int d^4\tau \delta^4(x - x(\tau)) \frac{\rho^i \rho^j}{\sqrt{\rho^2}} \partial_i x^\mu \partial_j x^\nu, \quad (53)$$

⁹ The action functional for the coupling of branes to (non)-Abelian gauge fields is suggested in [25].

¹⁰ A group-theoretical approach to hydrodynamics can be found, for example, in [26], where the equations of motion of a fluid are regarded as the equations of geodesics on the infinite-dimensional group of diffeomorphisms generated by the fluid flows, which is equipped with a right-invariant Riemannian metric.

¹¹ Notice that the matter action (49) can be easily generalized to the case of a perfect fluid with pressure (see for other Lagrangians [27, 28, 29, 30] and references therein). Introducing a scalar density $p(\tau)$ on the brane N we define the action functional as

$$S[x(\tau)] = - \int_N d^4\tau \left[\sqrt{\rho^2} + p \ln \sqrt{\frac{\rho^2}{-h}} \right],$$

where h is the determinant of the induced metric, the vector density $\rho^i(\tau)$ is the internal energy flow and $p(\tau)$ characterizes the fluid pressure.

that is the energy-momentum tensor for a relativistic dust (see, e.g., [31]).

In a certain sense the hydrodynamical approach is reduced to the particle one under the assumption

$$e^i(\tau) = \lambda(\tau) \rho^i(\tau), \quad (54)$$

where $\lambda(\tau)$ is some scalar function on N describing the charge to mass ratio. If that is the case Eqs. (52) give rise to

$$\partial_i \rho^i = 0, \quad \rho^i \partial_i \lambda = 0, \quad (55)$$

i.e. to the mass conservation law. Provided the requirement (54) is fulfilled the equations of motion (52) possess a “partial” reparametrization invariance, which implies orthogonality of the equations to $\rho^i \partial_i x^\mu$.

Now bearing in mind the assumption (54) we proceed to multipoles. Let $z^\mu(\sigma)$ be a naturally parametrized worldline in Minkowski space, then similarly to (6) we claim that

$$\dot{z}_\rho(\tau^0) \xi^\rho(\tau) = 0, \quad \dot{z}^2(\tau^0) = 1, \quad (56)$$

where $\xi^\mu(\tau) = x^\mu(\tau) - z^\mu(\tau^0)$ and overdots denote the derivative with respect to τ^0 . The condition (56) partially specifies a parametrization on the brane N and foliates it into a family of the space-like hypersurfaces $\tau^0 = \text{const}$. Thereby the condition (56) fixes the gauge and spoils the reparametrization invariance of the equations of motion. The trajectory $z^\mu(\tau^0)$ of the center of mass is defined by

$$\int d\bar{\tau} \rho^0(\tau) \xi^\mu(\tau) = 0. \quad (57)$$

The definition of multipoles (9) transforms into

$$\int d\bar{\tau} e^0(\tau) \xi_{\mu_1}(\tau) \dots \xi_{\mu_n}(\tau), \quad (58)$$

$$\int d\bar{\tau} e^i(\tau) \xi_{\mu_1}(\tau) \dots \xi_{[\mu_{n-1}}(\tau) \partial_i \xi_{\mu_n]}(\tau) \text{pr}_{\mu_n}^\rho.$$

It is easy to see that the Liénard-Wiechert potentials are expressed in terms of the multipole moments (58) and their derivatives. Indeed, expanding in powers of ξ the δ -function in the formula (5) adjusted to the hydrodynamical case we obtain expressions containing $\xi_\mu(\tau)$ in the following way

$$\int d\bar{\tau} \delta^{(k)}((x - z(\tau^0))^2) \times \xi_{\mu_1}(\tau) \dots \xi_{\mu_{n-1}}(\tau) e^i(\tau) \partial_i (z_{\mu_n}(\tau^0) + \xi_{\mu_n}(\tau)), \quad (59)$$

that are obviously expressed in terms of the multipole moments (58).

As to mechanical moments are concerned the equations of motion for particles (3) suggest a general corresponding rule between mechanical moments in the particle approach and hydrodynamical one

$$\sum_a \leftrightarrow \int d\bar{\tau}, \quad m_a \leftrightarrow \rho^0, \quad d/d\tau \leftrightarrow \rho^i/\rho^0 \partial_i. \quad (60)$$

In our case the ambiguity of this rule must be settled by the prescription that, at first, masses m_a are removed from under the differentiations with respect to τ . For instance, the intrinsic stress tensor arising in (31) is rewritten as

$$T_{\mu\nu} = \int d\bar{\tau} \frac{\rho^i \rho^j}{\rho^0} \partial_i \xi_\rho \partial_j \xi_\sigma \text{pr}_\mu^\rho \text{pr}_\nu^\sigma. \quad (61)$$

In the sequel we do not repeat calculations of the preceding section, but we show that the resulting equations of motion of the effective model are identical to ones derived in the particle framework. Besides, in spite of this assertion the equations of motion for the center of mass (23) were rederived by direct calculations.

Similarly to the considerations regarding the Liénard-Wiechert potentials it can be proved that, firstly, the equations of motion for the center of mass are expressed in terms of the mechanical moments and multipoles (58); secondly, on introducing species¹² the evolution equations for the intrinsic species moments both mechanical and electric (magnetic) are expressed in terms of the intrinsic species moments. In the hydrodynamical framework the species are domains on the hypersurfaces $\tau^0 = \text{const}$ of the constant ratio $\lambda(\tau^0, \bar{\tau})$. The form of these domains depends on τ^0 to be consistent with (55).

Whereas the equations of motion and expressions for multipoles in the hydrodynamical case pass into respective expressions for the particle case with the assumption

$$\rho^i(\tau) = (\rho^0(\bar{\tau}), \mathbf{0}), \quad (62)$$

we infer that the above-mentioned evolution equations are identical to the equations derived in the preceding section. Notice that the mass renormalization (22) in the hydrodynamical framework looks like

$$\tilde{\rho}^i = \rho^i + \frac{e^i q}{2} \varepsilon^{-\frac{1}{2}} \Rightarrow \partial_i \tilde{\rho}^i = 0. \quad (63)$$

In conclusion we point out that the elaborated approach to the definition of the multipole moments is a quite general one and can be used to describe the multipole moments for a charged fluid or system of charged particles approximated by not only a point particle but also a brane, i.e. to define in a Poincaré-invariant way the intrinsic linear density of dipole moment for a string or the intrinsic surface density of magnetic moment for a membrane etc.

Let us split the coordinates $\{\tau^i\}$ on the brane N into $\{\tau^a, \bar{\tau}^a\}$, τ^0 is included to $\{\tau^a\}$. Suppose given a brane

M with coordinates $\{\sigma^a\}$ that is embedded into the space-time by a smooth mapping $z^\mu(\sigma)$. The induced metric on the brane M we denote by h_{ab} . Suppose the brane N is also submersed into M by the mapping

$$\varphi : N \rightarrow M, \quad \sigma^a = \varphi^a(\tau, \bar{\tau}) = \tau^a, \quad (64)$$

i.e. the mapping φ matches systems of coordinates on the branes N and M . For our choice it merely identifies the coordinates $\{\tau^a\}$ and $\{\sigma^a\}$. On the one hand this mapping foliates the brane N into a family of the space-like surfaces $\tau^a = \text{const}$. On the other hand it foliates the brane N and, consequently, the space-time into a family of surfaces diffeomorphic to the brane M . Then we claim that the vector field $(\rho^b \rho^c h_{bc})^{-\frac{1}{2}} \rho^a \partial_a$ is projectable onto leaves of the latter foliation, i.e.

$$[\partial_{\bar{a}}, \frac{\rho^b}{\sqrt{\rho^2}} \partial_b] = 0, \quad (65)$$

hereinafter $\rho^2(\tau) = \rho^b(\tau) \rho^c(\tau) h_{bc}(\sigma)$. Evidently this requirement can be satisfied by appropriately chosen $\rho^i(\tau)$ and $x^\mu(\tau)$ for any given vector density

$$V^\mu(x) = \int_N d^4 \tau \delta^4(x - x(\tau)) \rho^i(\tau) \partial_i x^\mu(\tau), \quad (66)$$

on the space-time.

The transverse condition (56) becomes

$$\partial_a z_\rho(\sigma) \xi^\rho(\tau) = 0, \quad (67)$$

where $\xi^\mu(\tau) = x^\mu(\sigma, \bar{\tau}) - z^\mu(\sigma)$. This condition partially fixes a system of coordinates on the brane N , i.e. it specifies the space-like surfaces $\tau^a = \text{const}$ on the brane. The center of mass condition modifies into

$$\int d\bar{\tau} \sqrt{\rho^2(\tau)} \xi^\mu(\tau) = 0 \Leftrightarrow \int d\bar{\tau} \rho^a(\tau) \xi^\mu(\tau) = 0, \quad (68)$$

and the intrinsic electric and magnetic multipole moments are defined by

$$\int d\bar{\tau} e^a(\tau) \xi_{\mu_1}(\tau) \dots \xi_{\mu_n}(\tau), \quad (69)$$

$$\frac{\rho^a}{\sqrt{\rho^2}} \int d\bar{\tau} e^i(\tau) \xi_{\mu_1}(\tau) \dots \xi_{[\mu_{n-1}}(\tau) \partial_i \xi_\rho(\tau) \text{pr}_{\mu_n]}^\rho,$$

where $\text{pr}_\mu^\nu = \delta_\mu^\nu - \rho^a \rho^b \partial_a z^\nu \partial_b z_\mu / \rho^2$ and we assume the relation (54) is fulfilled. The mechanical moments are generalized in an obvious manner. Clearly, the so defined multipoles both mechanical and electric (magnetic) are orthogonal to $\partial_a z^\mu$ and zeroth multipoles are conserved

$$\partial_a \int d\bar{\tau} e^a(\tau) = \partial_a \int d\bar{\tau} \rho^a(\tau) = 0, \quad (70)$$

as long as corresponding currents are conserved.

¹² We emphasize once more that separation into species is the simplest means to obtain a system of ordinary differential equations from the integro-differential one reducing to the multipole moments (58) only. For example, another interesting choice of the ratio λ is to assume it to be the kernel of a differential operator compatible with (55).

V. CONCLUDING REMARKS

Let us summarize the main results of our research. We have investigated the effective dynamics of a system of charged particles within two approaches – the particle itself and hydrodynamical one. We have described the effective dynamics of such an object by means of a system of ordinary differential evolution equations for the intrinsic multipole moments and the center of mass. We have proved the equivalence of two examined approaches to the problem of radiation reaction for multipole moments. In passing we have derived the effective model for a neutral pointlike system of charged particles.

These results can be extended to several directions. We point out some of them only. It would be interesting to study the effective dynamics of a string or membrane with intrinsic higher multipole moments like that is given in [12] for an electrically charged string. Another direction for further research could be in a study of a generalization of the effective dynamics of a high-current beam

of charged particles to include the matter term proposed in Section IV. Other points could be the nonrelativistic effective dynamics of a neutral pointlike object with dipole moment or may be the effective dynamics of a neutral pointlike object with magnetic moment and vanishing electric dipole moment subjected to some simplifying assumptions.

Acknowledgments

I am grateful to I.V. Gorbunov and A.A. Sharapov for carefully reading the draft of this manuscript and for their constructive criticism. This work was supported by the RFBR grant 06-02-17352 and the grant for Support of Russian Scientific Schools SS-5103.2006.2. Author appreciates financial support from the Dynasty Foundation and International Center for Fundamental Physics in Moscow.

-
- [1] H.A. Lorentz, *Theory of Electrons* (B.G. Teubner, Leipzig, 1909).
 - [2] P.A.M. Dirac, *Classical theory of radiating electrons*, Proc. Roy. Soc. London A **167**, 148 (1938).
 - [3] B.S. DeWitt, R.W. Brehme, *Radiation damping in a gravitational field*, Ann. Phys. **9**, 220 (1960).
 - [4] B.P. Kosyakov, *Exact solutions of classical electrodynamics and the Yang-Mills-Wong theory in even-dimensional spacetime*, Teor. Mat. Fiz. **119**, 119 (1999) (in Russian). [Theor. Math. Phys. **199**, 493 (1999)]. hep-th/0207217.
 - [5] P.O. Kazinski, S.L. Lyakhovich, and A.A. Sharapov, *Radiation reaction and renormalization in classical electrodynamics of a point particle in any dimension*, Phys. Rev. D **66**, 025017 (2002). hep-th/0201046.
 - [6] J. Frenkel, *Die Elektrodynamik des rotierenden Elektrons*, Z. Physik **37**, 243 (1926).
 - [7] H.J. Bhabha and H.C. Corben, *General classical theory of spinning particles in a Maxwell field*, Proc. Roy. Soc. London A **178**, 273 (1941).
 - [8] V. Bargmann, L. Michel, and V.L. Telegdi, *Precession of the polarization of particles moving in a homogeneous electromagnetic field*, Phys. Rev. Lett. **2**, 435 (1959).
 - [9] P.E.G. Rowe, G.T. Rowe, *The classical equations of motion for a spinning particle with charge and magnetic moment*, Phys. Rep. **149**, 287 (1987).
 - [10] S.L. Lyakhovich, A.Yu. Segal, and A.A. Sharapov, *A universal model of $D=4$ spinning particle*, Phys. Rev. D **54**, 5223 (1996). hep-th/9603174.
 - [11] A. Frydryszak, *Lagrangian models of particles with spin: the first seventy years*, hep-th/9601020.
 - [12] P.O. Kazinski, *Effective dynamics of an electrically charged string with a current*, Zh. Eksp. Teor. Fiz. **128**, 312 (2005) (in Russian). [J. Exp. Theor. Phys. **101**, 270 (2005)]. hep-th/0507237.
 - [13] A. Ori and E. Rosenthal, *Calculation of the self force using the extended object approach*, J. Math. Phys. **45**, 2347 (2004). gr-qc/0309102.
 - [14] L.D. Landau and E.M. Lifshitz, *The Classical Theory of Fields* (Pergamon, Oxford, 1962).
 - [15] N. Itoh, *Radiation reaction due to magnetic dipole radiation*, Phys. Rev. A **43**, 1002 (1991).
 - [16] S.-K. Ma, *Modern Theory of Critical Phenomena* (W.A. Benjamin Inc. Advanced Book Program Reading, Massachusetts, 1976).
 - [17] R. Courant and D. Hilbert, *Methods of Mathematical Physics Vol. II: Partial Differential Equations* (John Wiley & Sons, New York, London, 1962).
 - [18] I.M. Gel'fand and G.E. Shilov, *Generalized Functions Vol. I: Properties and Operations* (Academic Press, New York, London, 1964).
 - [19] P.O. Kazinski and A.A. Sharapov, *Radiation reaction for a massless charged particle*, Class. Quant. Grav. **20**, 2715 (2003). hep-th/0212286.
 - [20] P.O. Kazinski and A.A. Sharapov, *Radiation back-reaction and renormalization in classical field theory with singular sources*, Teor. Mat. Fiz. **143**, 375 (2005) (in Russian). [Theor. Math. Phys. **143**, 798 (2005)].
 - [21] E.S. Fradkin, *On reaction of the self-field of a charged particle*, Zh. Eksp. Teor. Fiz. **20**, 211 (1950) (in Russian).
 - [22] A.O. Barut, *Electrodynamics and Classical Theory of Fields and Particles* (Dover, New York, 1964).
 - [23] V.G. Bagrov, G.S. Bisnovatyi-Kogan, V.A. Borodovitsyn, A.V. Borisov, O.F. Dorofeev, V.Ch. Zhukovskiy, Yu.L. Pivovarov, O.V. Shorokhov, and V.Ya. Epp, *Theory of Radiation of Relativistic Particles*, (Fiz. Mat. Lit., Moscow, 2002) (in Russian).
 - [24] D. Bohm and M. Weinstein, *The self-oscillations of a charged particle*, Phys. Rev. **74**, 1789 (1948).
 - [25] A.P. Balachandran, B.S. Skagerstam, and A. Stern, *Gauge theory of extended objects*, Phys. Rev. D **74**, 439 (1979).
 - [26] V.I. Arnold and B.A. Khesin, *Topological Methods in Hydrodynamics* (Springer, New York, 1998).
 - [27] A.H. Taub, *General relativistic variational principle for*

- perfect fluids*, Phys. Rev. **94**, 1468 (1954).
- [28] B.F. Schutz, Jr., *Perfect fluids in general relativity: velocity potentials and a variational principle*, Phys. Rev. D **2**, 2762 (1970).
- [29] D. Brown, *Action functionals for relativistic perfect fluids*, Class. Quant. Grav. **10**, 1579 (1993). gr-qc/9304026.
- [30] P. Hájíček and J. Kijowski, *Lagrangian and Hamiltonian formalism for discontinuous fluid and gravitational field*, Phys. Rev. D **57**, 914 (1998); **61** 129901(E) (2000) (erratum). gr-qc/9707020.
- [31] L.D. Landau and E.M. Lifshitz, *Fluid Mechanics*, (Pergamon, Oxford, 1987).